

ESTIMATES ON SINGULAR VALUES OF FUNCTIONS OF PERTURBED OPERATORS

QINBO LIU

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY
EAST LANSING, MI 48824, USA

ABSTRACT. This is a continuation of [1] and [2]. We prove that if function f belongs to the class $\Lambda_\omega \stackrel{\text{def}}{=} \{f : \omega_f(\delta) \leq \text{const } \omega(\delta)\}$ for an arbitrary modulus of continuity ω , then $s_j(f(A) - f(B)) \leq c \cdot \omega_*((1 + j)^{-\frac{1}{p}} \|A - B\|_{S_p^l}) \cdot \|f\|_{\Lambda_\omega}$ for arbitrary self-adjoint operators A, B and all $1 \leq j \leq l$, where $\omega_*(x) \stackrel{\text{def}}{=} x \int_x^\infty \frac{\omega(t)}{t^2} dt$ ($x > 0$). The result is then generalized for contractions, maximal dissipative operators, normal operators and n -tuples of commuting self-adjoint operators.

CONTENTS

1. Introduction	1
2. Space Λ_ω	2
3. Estimates on singular values of functions of perturbed self-adjoint and unitary operators	3
4. Estimates for other types of operators	8
References	9

1. INTRODUCTION

In this note we study the behavior of functions of operators under perturbations. We are going to find estimates for the singular values $s_n(f(A) - f(B))$, where both A and B are arbitrary self-adjoint or unitary operators. These results are based on the methods developed in [1] and [3] for estimates of operator norms $\|f(A) - f(B)\|$, in these papers the authors proved if f belongs to the Hölder class $\Lambda_\alpha(\mathbb{R})$ with $0 < \alpha < 1$, then $\|f(A) - f(B)\| \leq \text{const } \|f\|_{\Lambda_\alpha} \|A - B\|^\alpha$ for all pairs of self-adjoint or unitary operators A and B . The authors also generalized their results to the class Λ_ω , and obtained estimate $\|f(A) - f(B)\| \leq \text{const } \|f\|_{\Lambda_\omega} \omega_* \|A - B\|$.

In [2], it was shown that for functions f in the Hölder class $\Lambda_\alpha(\mathbb{R})$ with $0 < \alpha < 1$ and if $1 < p < \infty$, the operator $f(A) - f(B)$ belongs to $\mathbf{S}_{p/\alpha}$, whenever A and B are arbitrary self-adjoint operators such that $A - B \in \mathbf{S}_p$.

Date: April 25, 2016.

Corresponding Author:

Email address: liuqinbo@msu.edu(Qinbo Liu).

In particular, it was proved that if $0 < \alpha < 1$, then there exists a constant $c > 0$ such that for every $l \geq 0$, $p \in [1, \infty)$, $f \in \Lambda_\alpha(\mathbb{R})$, and for arbitrary self-adjoint operators A and B on Hilbert space with bounded $A - B$, the following inequality holds for every $j \leq l$:

$$s_j(f(A) - f(B)) \leq c \|f\|_{\Lambda_\alpha(\mathbb{R})} (1+j)^{-\frac{\alpha}{p}} \|A - B\|_{S_p^l}^\alpha \text{ (see (3.1) for definition).}$$

In section §3, we generalize this estimate to the class Λ_ω and also obtain some lower-bound estimates for rank one perturbations which also extend the results in [2]. In section §4, similar estimates are given without proofs in case of contractions, maximal dissipative operators, normal operators and n -tuples of commuting self-adjoint operators.

Necessary information on Space Λ_ω is given in section §2. We refer the reader to [1] for more detailed information.

2. SPACE Λ_ω

Let ω be a modulus of continuity, i.e., ω is a nondecreasing continuous function on $[0, \infty)$ such that $\omega(0) = 0$, $\omega(x) > 0$ for $x > 0$, and

$$\omega(x+y) \leq \omega(x) + \omega(y), x, y \in [0, \infty).$$

We denote by $\Lambda_\omega(\mathbb{R})$ the space of functions on \mathbb{R} such that

$$\|f\|_{\Lambda_\omega(\mathbb{R})} \stackrel{\text{def}}{=} \sup_{x \neq y} \frac{|f(x) - f(y)|}{\omega(|x - y|)}.$$

The space $\Lambda_\omega(\mathbb{T})$ on the unit circle can be defined in a similar way.

We continue with the class Λ_ω of functions on \mathbb{T} first. Let w be an infinitely differentiable function on \mathbb{R} such that

$$w \geq 0, \text{ supp } w \subset [\frac{1}{2}, 2], \text{ and } w(x) = 1 - w(\frac{x}{2}) \text{ for } x \in [1, 2]. \quad (2.1)$$

Define a C^∞ function v on \mathbb{R} by

$$v(x) = 1 \text{ for } x \in [-1, 1] \text{ and } v(x) = w(|x|) \text{ if } |x| \geq 1. \quad (2.2)$$

Define trigonometric polynomials W_n , W_n^\sharp and V_n by

$$W_n(z) = \sum_{k \in \mathbb{Z}} w(\frac{k}{2^n}) z^k, n \geq 1, W_0(z) = \bar{z} + 1 + z, \text{ and } W_n^\sharp(z) = \overline{W_n(z)}, n \geq 0$$

and

$$V_n(z) = \sum_{k \in \mathbb{Z}} w(\frac{k}{2^n}) z^k, n \geq 1.$$

V_n is called de la Vallée Poussin type kernel.

If f is a distribution on \mathbb{T} , we define f_n , $n \geq 0$ by

$$f_n = f * W_n + f * W_n^\sharp, n \geq 1, \text{ and } f_0 = f * W_0,$$

Then $f = \sum_{n \geq 0} f_n$ and $f - f * V_n = \sum_{k=n+1}^\infty f_k$.

Now we proceed to the real line case. We use the same functions w, v as in (2.1), (2.2), and define functions W_n, W_n^\sharp and V_n on \mathbb{R} by

$$\mathcal{F}W_n(x) = w\left(\frac{x}{2^n}\right), \mathcal{F}W_n^\sharp(x) = \mathcal{F}W_n(-x), n \in \mathbb{Z}$$

and

$$\mathcal{F}V_n(x) = v\left(\frac{x}{2^n}\right), n \in \mathbb{Z},$$

where \mathcal{F} is the *Fourier transform*:

$$(\mathcal{F}f)(t) = \int_{\mathbb{R}} f(x)e^{-ixt}dx, f \in L^1.$$

V_n is also called de la Vallée Poussin type kernel.

If f is a tempered distribution on \mathbb{R} , we define f_n by

$$f_n = f * W_n + f * W_n^\sharp, n \in \mathbb{Z}.$$

We will use the same notation $\Lambda_\omega, W_n, W_n^\sharp$ and V_n on \mathbb{R} and on \mathbb{T} in the following discussion.

In [1], it is proved that there exists a constant c such that for an arbitrary modulus of continuity ω and for an arbitrary function f in Λ_ω , the following inequalities hold for all $n \in \mathbb{Z}$, in \mathbb{R} case, or for all $n \geq 0$, in \mathbb{T} case:

$$\|f - f * V_n\|_{L^\infty} \leq c \omega(2^{-n}) \|f\|_{\Lambda_\omega} \quad (2.3)$$

$$\|f * W_n\|_{L^\infty} \leq c \omega(2^{-n}) \|f\|_{\Lambda_\omega}, \|f * W_n^\sharp\|_{L^\infty} \leq c \omega(2^{-n}) \|f\|_{\Lambda_\omega} \quad (2.4)$$

Let $\mathcal{S}'(\mathbb{R})$ be the space of all tempered distributions on \mathbb{R} . Denote by $\mathcal{S}'_+(\mathbb{R})$ the set of all $f \in \mathcal{S}'(\mathbb{R})$ such that $\text{supp } \mathcal{F}f \subset [0, \infty)$. Put $(\Lambda_\omega(\mathbb{R}))_+ \stackrel{\text{def}}{=} \Lambda_\omega(\mathbb{R}) \cap \mathcal{S}'_+(\mathbb{R})$ and $\mathbb{C}_+ \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \text{Im } z > 0\}$. Then a function in $\Lambda_\omega(\mathbb{R})$ belongs to the space $(\Lambda_\omega(\mathbb{R}))_+$ if and only if it has a (unique) continuous extension to the closed upper half-plane $\text{clos } \mathbb{C}_+$ that is analytic in the open upper half-plane \mathbb{C}_+ with at most a polynomial growth rate at infinity.

3. ESTIMATES ON SINGULAR VALUES OF FUNCTIONS OF PERTURBED SELF-ADJOINT AND UNITARY OPERATORS

Recall that if T is a bounded linear operator on Hilbert space, then the singular values $s_j(T)$, $j \geq 0$, are defined by

$$s_j(T) = \inf\{\|T - R\| : \text{rank } R \leq j\}.$$

For $l \geq 0$ and $p \geq 1$, we consider the norm S_p^l (see [9]) defined by

$$\|T\|_{S_p^l} \stackrel{\text{def}}{=} \left(\sum_{j=0}^l (s_j(T))^p \right)^{\frac{1}{p}}. \quad (3.1)$$

It is shown in [13] and [2] that if f is an entire function of exponential type at most σ that is bounded on \mathbb{R} , and A, B are self-adjoint operators with bounded $A - B$, then

$$\|f(A) - f(B)\| \leq \text{const } \sigma \|f\|_{L^\infty} \|A - B\|, \quad (3.2)$$

and

$$\|f(A) - f(B)\|_{S_p^l} \leq \text{const } \sigma \|f\|_{L^\infty} \|A - B\|_{S_p^l}. \quad (3.3)$$

For the proof and more details, see [1], [2], [8], [10], [11] and [13].

Given a modulus of continuity ω , define functions ω_* and ω_\sharp by

$$\omega_*(x) = x \int_x^\infty \frac{\omega(t)}{t^2} dt, \quad x > 0$$

and

$$\omega_\sharp(x) = x \int_x^\infty \frac{\omega(t)}{t^2} dt + \int_0^x \frac{\omega(t)}{t} dt, \quad x > 0.$$

In this paper, we assume that ω_\sharp is finite valued whenever it is used. For example, if we define ω by

$$\omega(x) = x^\alpha, \quad x > 0, \quad 0 < \alpha < 1,$$

then $\omega_\sharp(x) \leq \text{const } \omega(x)$.

It is well known (see [6], Ch.3, Theorem 13.30) that if ω is a modulus of continuity, then the Hilbert transform maps Λ_ω into itself if and only if $\omega_\sharp(x) \leq \text{const } \omega(x)$.

Theorem 3.1. *There exists a constant $c > 0$ such that for every modulus of continuity ω , every f in $\Lambda_\omega(\mathbb{R})$ and for arbitrary self-adjoint operators A and B , the following inequality holds for all l and for all j , $1 \leq j \leq l$:*

$$s_j(f(A) - f(B)) \leq c \cdot \omega_*((1+j)^{-\frac{1}{p}} \|A - B\|_{S_p^l}) \cdot \|f\|_{\Lambda_\omega}. \quad (3.4)$$

Proof. A and B can be taken as bounded operators (see [3], Lemma 4.4), then we may further assume f is bounded. Let $R_N = \sum_{n=-\infty}^N (f_n(A) - f_n(B))$, $Q_N = (f - f * V_N)(A) - (f - f * V_N)(B)$. Here f_n and the de la Vallée Poussin type kernel V_N are defined as in §2. Then $f(A) - f(B) = R_N + Q_N$, with convergence in the uniform operator topology as shown in [1]. Note that for any integer $m \in \mathbb{Z}$, functions f_m and $f - f * V_m$ are entire functions of exponential type at most 2^{m+1} . Thus it follows from (3.2), (3.3), (2.3), and (2.4) that

$$\|Q_N\| \leq c \cdot \omega(2^{-N}) \cdot \|f\|_{\Lambda_\omega},$$

and

$$\begin{aligned}
\|R_N\|_{S_p^l} &\leq \sum_{n=-\infty}^N \|f_n(A) - f_n(B)\|_{S_p^l} \\
&\leq c \cdot \sum_{n=-\infty}^N (2^n \cdot \|f_n\|_{L^\infty}) \cdot \|A - B\|_{S_p^l} \\
&\leq c \cdot 2^N \cdot \omega_*(2^{-N}) \cdot \|A - B\|_{S_p^l} \cdot \|f\|_{\Lambda_\omega} \text{ (see [1])}
\end{aligned}$$

Then

$$\begin{aligned}
s_j(f(A) - f(B)) &\leq s_j(R_N) + \|Q_N\| \leq (1+j)^{-1/p} \cdot \|R_N\|_{S_p^l} + \|Q_N\| \\
&\leq c \cdot [(1+j)^{-\frac{1}{p}} \cdot 2^N \cdot \omega_*(2^{-N}) \|A - B\|_{S_p^l} + \omega(2^{-N})] \\
&\quad \cdot \|f\|_{\Lambda_\omega}
\end{aligned}$$

Take N such that $1 \leq (1+j)^{-\frac{1}{p}} \cdot 2^N \cdot \|A - B\|_{S_p^l} \leq 2$ and use the fact that $\omega(t) \leq \omega_*(t)$ for any $t > 0$, we get (3.4). \square

Theorem 3.2. *There exists a constant $c > 0$ such that for every modulus of continuity ω , every f in $\Lambda_\omega(\mathbb{T})$ and for arbitrary unitary operators U and V , the following inequality holds for all l and for all j , $1 \leq j \leq l$:*

$$s_j(f(U) - f(V)) \leq c \cdot \omega_*((1+j)^{-\frac{1}{p}} \|U - V\|_{S_p^l}) \cdot \|f\|_{\Lambda_\omega}. \quad (3.5)$$

Proof. If $(1+j)^{-\frac{1}{p}} \cdot \|U - V\|_{S_p^l} \leq 2$, the proof is similar to Theorem 3.1 with $R_N = \sum_{n=0}^N (f_n(U) - f_n(V))$; if $(1+j)^{-\frac{1}{p}} \cdot \|U - V\|_{S_p^l} > 2$, then

$$s_j(f(U) - f(V)) \leq \|f(U) - f(V)\| \leq c \cdot \omega_*(\|U - V\|) \cdot \|f\|_{\Lambda_\omega} \leq c \cdot \omega_*(2) \cdot \|f\|_{\Lambda_\omega}.$$

\square

Corollary 3.3. *Let ω be a modulus of continuity such that*

$$\omega_*(x) \leq \text{const } \omega(x), \quad x \geq 0.$$

Then for an arbitrary function $f \in \Lambda_\omega(\mathbb{R})$ and for arbitrary self-adjoint operators A and B , the following inequality holds for all l and for all j , $1 \leq j \leq l$:

$$s_j(f(A) - f(B)) \leq \text{const } \omega((1+j)^{-\frac{1}{p}} \|A - B\|_{S_p^l}) \|f\|_{\Lambda_\omega}.$$

Let H, \mathcal{H} be the Hankel operators defined in [2].

Theorem 3.4. *Let ω be a modulus of continuity on \mathbb{T} . There exist unitary operators U, V and a real function h in $\Lambda_{\omega_{\sharp}}((T))$ such that*

$$\text{rank}(U - V) = 1 \quad \text{and} \quad s_m(h(U) - h(V)) \geq \omega((1+m)^{-1}).$$

Proof. Consider the operators U and V on space $L_2(\mathbb{T})$ with respect to the normalized Lebesgue measure on \mathbb{T} defined by (see [2])

$$Uf = \bar{z}f \text{ and } Vf = \bar{z}f - 2(f, 1)\bar{z}, \quad f \in L^2.$$

For $f \in C(\mathbb{T})$, we have

$$((f(U) - f(V))z^j, z^k) = -2 \begin{cases} \hat{f}(j - k), & \text{if } j \geq 0, k < 0; \\ \hat{f}(j - k), & \text{if } j < 0, k \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Define function g by

$$g(\zeta) = \sum_{n=1}^{\infty} \omega(4^{-n})(\zeta^{4^n} + \bar{\zeta}^{4^n}), \quad \zeta \in \mathbb{T}.$$

Then we have

$$\|g * W_n\|_{L^\infty} \leq \text{const } \omega(2^{-n}), \quad \|g * W_n^\sharp\|_{L^\infty} \leq \text{const } \omega(2^{-n}), \quad n \geq 0.$$

Let ξ, η be two arbitrarily different fixed points on \mathbb{T} , choose $N \geq 0$ such that $\frac{1}{2} \leq \frac{2^{-N}}{|\xi - \eta|} \leq 1$, then

$$\begin{aligned} |g(\xi) - g(\eta)| &\leq \sum_{n=0}^N |g_n(\xi) - g_n(\eta)| + |(g - g * V_N)(\xi) - (g - g * V_N)(\eta)| \\ &\leq \sum_{n=0}^N |g_n(\xi) - g_n(\eta)| + 2 \sum_{n=N+1}^{\infty} \|g_n\|_{L^\infty} \\ &\leq \text{const} \sum_{n=0}^N 2^n \|g_n\|_{L^\infty} |\xi - \eta| + 2 \sum_{n=N+1}^{\infty} \|g_n\|_{L^\infty} \\ &\leq \text{const} \sum_{n=0}^N 2^n \omega(2^{-n}) |\xi - \eta| + \text{const} \sum_{n=N+1}^{\infty} \omega(2^{-n}) \\ &\leq \text{const } \omega_*(|\xi - \eta|) + \text{const} \int_0^{2^{-N}} \frac{\omega(t)}{t} dt \\ &\leq \text{const } \omega_\sharp(|\xi - \eta|). \end{aligned}$$

Consider the matrix $\Gamma_g = \{\hat{g}(-j - k)\}_{j \geq 1, k \geq 0} = \{\hat{g}(j + k)\}_{j \geq 1, k \geq 0}$. Let $n \geq 1$. Define matrix $T_n = \{\hat{g}(j + k + 4^{n-1} + 1)\}_{0 \leq j, k \leq 3 \cdot 4^{n-1}}$, then

$$T_n = \begin{bmatrix} & & & \omega(4^{-n}) \\ & & \omega(4^{-n}) & \\ & & \ddots & \\ \omega(4^{-n}) & & & \end{bmatrix}.$$

If R is any matrix with the same size of T_n such that $\text{rank}(R) < 3 \cdot 4^{n-1}$, then $\|T_n - R\| \geq \omega(4^{-n})$. It follows that $s_j(T_n) \geq \omega(4^{-n})$ for $j < 3 \cdot 4^{n-1}$.

For each T_n , there is some orthogonal projection P_n such that $T_n = P_n \Gamma_g P_n$, hence $s_j(\Gamma_g) \geq s_j(T_n) \geq \omega(4^{-n})$ for all n and for all j , $j < 3 \cdot 4^{n-1}$. Thus for all $j \geq 0$, we have

$$s_j(\Gamma_g) \geq \omega\left(\frac{3}{16} \cdot (j+1)^{-1}\right) \geq \frac{3}{32} \cdot \omega((j+1)^{-1}).$$

To complete the proof, it suffices to take $h = \frac{32}{3}g$. \square

Corollary 3.5. *Let ω be a modulus of continuity such that*

$$\omega_{\sharp}(x) \leq \text{const } \omega(x), \quad 0 \leq x \leq 2.$$

There exist unitary operators U, V and a real function h in $\Lambda_{\omega}(T)$ such that

$$\text{rank}(U - V) = 1 \quad \text{and} \quad s_m(h(U) - h(V)) \geq \omega((1+m)^{-1}).$$

Theorem 3.6. *Let ω be a modulus of continuity on \mathbb{T} and f be a continuous function on \mathbb{T} . If for all unitary operators U and V , we have*

$$s_n(f(U) - f(V)) \leq \text{const } \omega((1+n)^{-\frac{1}{p}} \|U - V\|_{S_p}), \quad \text{for all } n \geq 0,$$

then $f \in \Lambda_{\omega}(\mathbb{T})$.

Proof. Let $\zeta, \eta \in \mathbb{T}$, we can select commuting unitary operators U and V such that $s_0(U - V) = s_1(U - V) = \dots = s_n(U - V) = |\zeta - \eta|$ and $s_k(U - V) = 0$, $k \geq n+1$. Then $s_n(f(U) - f(V)) = |f(\zeta) - f(\eta)|$, $\|U - V\|_{S_p} = (1+n)^{\frac{1}{p}} \cdot |\zeta - \eta|$. \square

Theorem 3.7. *Let ω be a modulus of continuity on \mathbb{R} and f be a continuous function on \mathbb{R} . If for all self-adjoint operators A and B , we have*

$$s_n(f(A) - f(B)) \leq \text{const } \omega((1+n)^{-\frac{1}{p}} \|A - B\|_{S_p}), \quad \text{for all } n \geq 0,$$

then $f \in \Lambda_{\omega}(\mathbb{R})$.

Proof. Similar to Theorem 3.6. \square

Theorem 3.8. *Let ω be a modulus of continuity over \mathbb{R} . There exist self-adjoint operators A, B , and a real function f in $\Lambda_{\omega_{\sharp}}(\mathbb{R})$ such that*

$$\text{rank}(A - B) = 1 \quad \text{and} \quad s_m(f(A) - f(B)) \geq \omega((1+m)^{-1}), \quad \text{for all } m \geq 0.$$

Proof. WLOG, we assume $\omega(t) = \omega(2)$, for all $t \geq 2$, that is, ω can be regarded as a modulus of continuity on \mathbb{T} .

We then choose a function (see [2], Lemma 9.6) $\rho \in C^{\infty}(\mathbb{T})$ such that $\rho(\zeta) + \rho(i\zeta) = 1$, $\rho(\zeta) = \rho(\bar{\zeta})$ for all $\zeta \in \mathbb{T}$, and ρ vanishes in a neighborhood of the set $\{-1, 1\}$. Note that $\rho \in \Lambda_{\omega}(\mathbb{T})$, since $\omega(st) \geq \frac{s}{2}\omega(t)$, for all $t \geq 0$ and s , $0 < s < 1$.

Define function g_1 by

$$g_1(\zeta) = \sum_{n=1}^{\infty} \omega(4^{-n}) (\zeta^{4^n} + \bar{\zeta}^{4^n}), \quad \zeta \in \mathbb{T}.$$

Then $g_1 \in \Lambda_{\omega_{\sharp}}(\mathbb{T})$. If $g_0 \stackrel{\text{def}}{=} C\rho g_1$ for a sufficient large number C , then $g_0 \in \Lambda_{\omega_{\sharp}}(\mathbb{T})$, vanishes in a neighborhood of the set $\{-1, 1\}$ and $g_0(\zeta) = g_0(\bar{\zeta})$ for all $\zeta \in \mathbb{T}$, and $s_m(H_{g_0}) \geq \omega((1+m)^{-1})$ for all $m \geq 0$.

Define $\varphi(x) = (x^2 + 1)^{-1}$ (as in [2], Theorem 9.9), then there exists a compactly supported real bounded function f such that $f(\varphi(x)) = g_0(\frac{x-i}{x+i})$ and a simple calculation shows that f belongs to $\Lambda_{\omega_{\sharp}}(\mathbb{R})$. Denote $L_e^2(\mathbb{R})$ the subspace of even functions in $L^2(\mathbb{R})$. Consider operators A and B on $L_e^2(\mathbb{R})$ defined by $A(g) = \mathbf{H}^{-1}M_{\varphi}\mathbf{H}(g)$ and $B(g) = \varphi g$, here \mathbf{H} is the Hilbert transform defined on $L^2(\mathbb{R})$ (see [2]) and M_{φ} is the multiplication by φ . Then $\text{rank}(A - B) = 1$, and we have

$$s_m(f(B) - f(A)) \geq \sqrt{2}s_m(\mathcal{H}_{f \circ \varphi}) = \sqrt{2}s_m(H_{g_0}) \geq \sqrt{2}\omega((1+m)^{-1}). \quad \square$$

4. ESTIMATES FOR OTHER TYPES OF OPERATORS

The following estimates are given without proofs in case of contractions, maximal dissipative operators, normal operators and n -tuples of commuting self-adjoint operators.

Theorem 4.1. *There exists a constant $c > 0$ such that for every modulus of continuity ω , every f in $(\Lambda_{\omega}(\mathbb{R}))_+$ and for arbitrary contractions T and R on Hilbert space, the following inequality holds for all l and for all j , $1 \leq j \leq l$:*

$$s_j(f(T) - f(R)) \leq c \omega_*((1+j)^{-\frac{1}{p}} \|T - R\|_{S_p^l}) \|f\|_{\Lambda_{\omega}}.$$

To prove this result, the following result is important (see [1], [2] and [12]):

There exists a constant c such that for arbitrary trigonometric polynomial f of degree n and for arbitrary contractions T and R on Hilbert space,

$$\|(f(T) - f(R))\|_{S_p} \leq c n \|f\|_{L^{\infty}} \|T - R\|_{S_p}.$$

Denote \mathcal{F} the *Fourier transform* on $L_1(\mathbb{R}^n)$, $n \geq 1$ by:

$$(\mathcal{F}f)(t) = \int_{\mathbb{R}^n} f(x) e^{-i(x,t)} dx, \text{ where}$$

$$x = (x_1, \dots, x_n), \quad t = (t_1, \dots, t_n), \quad (x, t) \stackrel{\text{def}}{=} x_1 t_1 + \dots + x_n t_n.$$

Theorem 4.2. *There exists a constant $c > 0$ such that for every modulus of continuity ω , every f in $(\Lambda_{\omega}(\mathbb{R}))_+$ and for arbitrary maximal dissipative operators L and M with bounded difference, the following inequality holds for all l and for all j , $1 \leq j \leq l$:*

$$s_j(f(L) - f(M)) \leq c \omega_*((1+j)^{-\frac{1}{p}} \|L - M\|_{S_p^l}) \|f\|_{\Lambda_{\omega}}.$$

To prove this result, the following result is important(see [4]):

There exists a constant $c > 0$ such that for every function f in $H^\infty(\mathbb{C}_+)$ with $\text{supp } \mathcal{F}f \subset [0, \sigma]$, $\sigma > 0$, and for arbitrary maximal dissipative operators L and M with bounded difference,

$$\|f(L) - f(M)\|_{S_p} \leq c \sigma \|f\|_{L^\infty} \|L - M\|_{S_p}.$$

Theorem 4.3. *There exists a constant $c > 0$ such that for every modulus of continuity ω , every f in $\Lambda_\omega(\mathbb{R}^2)$ and for arbitrary normal operators N_1 and N_2 , the following inequality holds for all l and for all j , $1 \leq j \leq l$:*

$$s_j(f(N_1) - f(N_2)) \leq c \omega_*((1+j)^{-\frac{1}{p}} \|N_1 - N_2\|_{S_p^l}) \|f\|_{\Lambda_\omega}.$$

To prove this result, the following result is important(see [5]):

There exists a constant $c > 0$ such that for every bounded continuous function f on \mathbb{R}^2 with

$$\text{supp } \mathcal{F}f \subset \{\zeta \in \mathbb{C} : |\zeta| \leq \sigma\}, \quad \sigma > 0,$$

and for arbitrary normal operators N_1 and N_2 ,

$$\|(f(N_1) - f(N_2))\|_{S_p} \leq c \sigma \|f\|_{L^\infty} \|N_1 - N_2\|_{S_p}.$$

Theorem 4.4. *Let n be a positive integer and $p \geq 1$. There exists a positive number c_n such that for every modulus of continuity ω , every f in $\Lambda_\omega(\mathbb{R}^n)$ and for arbitrary n -tuples of commuting self-adjoint operators (A_1, \dots, A_n) and (B_1, \dots, B_n) , the following inequality holds for all l and for all j , $1 \leq j \leq l$:*

$$s_j(f(A_1, \dots, A_n) - f(B_1, \dots, B_n)) \leq c_n \max_{1 \leq j \leq n} \omega_*((1+j)^{-\frac{1}{p}} \|A_j - B_j\|_{S_p^l}) \|f\|_{\Lambda_\omega}.$$

To prove this result, the following result is important(see [7]):

There exists a constant $c_n > 0$ such that for every bounded continuous function f on \mathbb{R}^n with

$$\text{supp } \mathcal{F}f \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \sigma\}, \quad \sigma > 0,$$

and for arbitrary n -tuples of commuting self-adjoint operators (A_1, \dots, A_n) and (B_1, \dots, B_n) ,

$$\|f(A_1, \dots, A_n) - f(B_1, \dots, B_n)\|_{S_p} \leq c_n \sigma \|f\|_{L^\infty} \max_{1 \leq j \leq n} \|A_j - B_j\|_{S_p}.$$

REFERENCES

1. A.B. Aleksandrov, V.V. Peller, Operator Hölder-Zygmund functions, Adv. Math. **224** (2010), 910–966.
2. A.B. Aleksandrov, V.V. Peller, Functions of operator under perturbation of class S_p , J.Func.Anal. **258** (2010), 3675–3724.
3. A.B. Aleksandrov, V.V. Peller, Functions of perturbed unbounded self-adjoint operators. Operator Bernstein type inequalities, Indiana Univ. Math. J. **59:4** (2010), 1451–1490.

4. A.B. Aleksandrov, V.V. Peller, Functions of perturbed dissipative operators, *Algebra i Analiz* **23** (2011), 9–51; translation in *St. Petersburg Math. J.* **23** (2012), 209–238.
5. A.B. Aleksandrov, V.V. Peller, D. Potapov and F. Sukochev, Functions of normal operators under perturbations, *Adv. Math.* **226** (2011), 5216–5251.
6. A. Zygmund, *Trigonometric series*, 2nd ed. Vols. I, II. Cambridge University Press, New York, 1959.
7. F.L. Nazarov, V.V. Peller, Functions of n -tuples of commuting self-adjoint operators, *J. Funct. Anal.* **266** (2014), 5398–5428.
8. M.S. Birman, M.Z. Solomyak, Double Stieltjes operator integrals. III, *Problems of Math. Phys., Leningrad. Univ.* **6** (1973), 27–53 (Russian).
9. M.S. Birman, M.Z. Solomyak, *Spectral theory of selfadjoint operators in Hilbert spaces, Mathematics and its Applications (Soviet Series)*, D. Reidel Publishing Co., Dordrecht, 1987.
10. V.V. Peller, Hankel operators of class S_p and their applications (rational approximation, Gaussian processes, the problem of majorizing operators), *Mat. Sbornik*, **113** (1980), 538–581 (Russian).
English Transl. in *Math. USSR Sbornik*, **41** (1982), 443–479.
11. V.V. Peller, Hankel operators in theory of perturbations of unitary and self-adjoint operators, *Funktsional. Anal. i Prilozhen.* **19:2** (1985), 37–51 (Russian).
English Transl. in *Funct. Anal. Appl.* **19** (1985), 111–123.
12. V.V. Peller, For which f does $A - B \in S_p$ imply that $f(A) - f(B) \in S_p$?, *Operator Theory*, Birkhauser, **24** (1987), 289–294.
13. V.V. Peller, Hankel operators in the perturbation theory of unbounded self-adjoint operators. *Analysis and partial differential equations*, 529–544, *Lecture Notes in Pure and Appl. Math.*, **122**, Dekker, New York, 1990.